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Einstein's interior field equations in elastic media

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Abstract. A general solution of the field equations of general relativity theory has been obtained for an elastic sphere of constant density. The equations have been solved under the condition that the pressure is zero at the boundary of the sphere and the density is constant inside this boundary. The present investigation represents an attempt to apply Rayner's theory of elasticity in general relativity for the construction of realistic models.

1. Introduction

Einstein's interior field equations in the case of elastic bodies have been defined by Rayner (1963) as follows:

$$R_{ij}-\tfrac{1}{2}Rg_{ij} = -\rho\lambda_i\lambda_j+\tfrac{1}{2}C_{ij}^{kl}(\tilde{g}_{kl}-\tilde{g}_{kl}^{0}) \equiv -T_{ij}, \tag{1.1}$$

where ρ is the proper density, $\tilde{g}_{ij} = g_{ij}+\lambda_i\lambda_j$, λ_i is the four velocity,

$$\rho = -G_{rs}\lambda^r\lambda^s \geqslant 0, \qquad g_{rs}\lambda^r\lambda^s = -1.$$

g_{ij} is of signature $(+\ +\ +\ -)$. C_{ij}^{kl} is an elastic tensor and has been defined as

$$C_{ijkl} = \tilde{g}_{kr}^{0}\tilde{g}_{ls}^{0}C_{ij}^{rs} \tag{1.2}$$

and the tensors g_{ij}, $\tilde{g}_{ij}^{0}$, λ^i, C_{ij}^{kl} satisfy following conditions:

(i) Time dependence and orthogonality conditions

$$\begin{aligned} £_\lambda C_{ijkl} &= 0 \\ £_\lambda \tilde{g}_{ij}^{0} &= 0, \end{aligned} \tag{1.3}$$

where $£_\lambda$ is the Lie derivative with respect to the vector field λ^i

$$\begin{aligned} C_{ijkl}\lambda^l &= 0 \\ \tilde{g}_{ij}^{0}\lambda^j &= 0. \end{aligned} \tag{1.4}$$

(ii) Symmetry conditions:

$$\begin{aligned} C_{ijkl} &= C_{jikl} = C_{ijlk} = C_{klij} \\ \tilde{g}_{ij}^{0} &= \tilde{g}_{ji}^{0}. \end{aligned} \tag{1.5}$$

(iii) The tensor

$$C_{AB} = C_{ijkl} \qquad (A = i,j; \quad B = k,l) \tag{1.6}$$

is a matrix of rank 6 and $\tilde{g}_{ij}^{0}$ is a positive semi-definite matrix of rank 3.

(iv) C_{ijkl} is isotropic so that it admits the representation of the form

$$C_{ijkl} = \nu \mathfrak{g}^0_{ij}\mathfrak{g}^0_{kl} + \mu(\mathfrak{g}^0_{ik}\mathfrak{g}^0_{jl} + \mathfrak{g}^0_{il}\mathfrak{g}^0_{jk}) \tag{1.7}$$

where ν and μ are scalars such that

$$\nu_{,j}\lambda^j = \mu_{,j}\lambda^j = 0. \tag{1.8}$$

2. Field equations and boundary conditions

We shall consider physical systems which are static as well as spherically symmetric. We can then write our line element in the standard form

$$ds^2 = +e^{\alpha}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 - e^{\beta}\,dt^2, \tag{2.1}$$

where α and β are functions of r only.

The field equations for the line element (2.1) are given by

$$T_1{}^1 = e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{2.2}$$

$$T_2{}^2 = e^{-\alpha}\left(\frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} + \frac{\beta' - \alpha'}{2r}\right), \tag{2.3}$$

$$T_2{}^2 = T_3{}^3, \tag{2.4}$$

$$T_4{}^4 = -e^{-\alpha}\left(\frac{\alpha'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2}. \tag{2.5}$$

where prime denotes differentiation with respect to r.

We assume that density is constant inside the sphere and gravitational potentials are continuous.

3. Solutions of the field equations

From (1.4),

$$\begin{aligned} &C_{ijkl}\lambda^l = 0 \quad \text{or} \quad C_{ijk4} = 0 \quad \text{since} \quad \lambda^i = (0,0,0,\lambda^4) \\ &\mathfrak{g}^0_{ij}\lambda^j = 0 \quad \text{or} \quad \mathfrak{g}^0_{i4}\lambda^4 = 0, \quad \text{that is,} \quad \mathfrak{g}^0_{i4} = 0. \end{aligned} \tag{3.1}$$

From (1.6)

$$|\mathfrak{g}^0_{ij}| = \begin{vmatrix} \mathfrak{g}^0_{11} & \mathfrak{g}^0_{12} & \mathfrak{g}^0_{13} & \mathfrak{g}^0_{14} \\ \mathfrak{g}^0_{21} & \mathfrak{g}^0_{22} & \mathfrak{g}^0_{23} & \mathfrak{g}^0_{24} \\ \mathfrak{g}^0_{31} & \mathfrak{g}^0_{32} & \mathfrak{g}^0_{33} & \mathfrak{g}^0_{34} \\ \mathfrak{g}^0_{41} & \mathfrak{g}^0_{42} & \mathfrak{g}^0_{43} & \mathfrak{g}^0_{44} \end{vmatrix} = 0. \tag{3.2}$$

Let Tx denote the tangent space of a point (x) of V_4 and Sx the subspace of Tx orthogonal to λ^i. Then Sx is the spatial rest frame of a particle of the elastic body at (x). The tensors $\mathfrak{g}^0_{ij}$ and $\mathfrak{g}_{ij}$ each define matrices for undeformed and deformed elastic bodies respectively in Sx.

In a comoving coordinate system, we can satisfy (1.2) to (1.5) by taking $\tilde{g}^0_{ij}$ and C^{kl}_{ij} as functions of space coordinates only. We regard them as known. ν and μ are scalars and we also regard them as known.

$\tilde{g}^0_{ij}$ is the metric for the undeformed elastic body and hence we take it as flat, namely

$$\tilde{g}^0_{11} = 1, \qquad \tilde{g}^0_{22} = r^2, \qquad \tilde{g}^0_{33} = r^2 \sin^2\theta. \tag{3.3}$$

From equation (1.8) we have

$$\nu = \nu(r), \qquad \mu = \mu(r).$$

The only nonzero surviving components of C_{ijkl} are C_{1111}, C_{1122}, C_{1133}, C_{2222}, C_{2233}, C_{3333} and their values are

$$\begin{aligned} C_{1111} &= \nu+2\mu, \\ C_{2222} &= (\nu+2\mu)r^2, \\ C_{3333} &= (\nu+2\mu)r^4 \sin^4\theta, \\ C_{1122} &= \nu r^2, \\ C_{1133} &= \nu r^2 \sin^2\theta, \\ C_{2233} &= \nu r^4 \sin^2\theta. \end{aligned} \tag{3.4}$$

Values of C^{rs}_{ij} are given by

$$\begin{aligned} &C^{11}_{11} = C^{22}_{22} = C^{33}_{33} = \nu+2\mu, \\ &C^{22}_{11} = \frac{\nu}{r^2}, \qquad C^{11}_{22} = \nu r^2, \\ &C^{33}_{11} = \frac{\nu}{r^2\sin^2\theta}, \qquad C^{11}_{33} = \nu r^2 \sin^2\theta, \\ &C^{33}_{22} = \nu\frac{1}{\sin^2\theta}, \qquad C^{22}_{33} = \nu \sin^2\theta. \end{aligned} \tag{3.5}$$

The nonvanishing components of the energy–momentum tensor are given by

$$\begin{aligned} T_{11} &= -\tfrac{1}{2}C^{11}_{11}(\tilde{g}_{11}-\tilde{g}^0_{11})-\tfrac{1}{2}C^{22}_{11}(\tilde{g}_{22}-\tilde{g}^0_{22})-\tfrac{1}{2}C^{33}_{11}(\tilde{g}_{33}-\tilde{g}^0_{33}) = \tfrac{1}{2}(\nu+2\mu)(-e^{\alpha}+1), \\ T_{22} &= -\tfrac{1}{2}C^{11}_{22}(\tilde{g}_{11}-\tilde{g}^0_{11})-\tfrac{1}{2}C^{22}_{22}(\tilde{g}_{22}-\tilde{g}^0_{22})-\tfrac{1}{2}C^{33}_{22}(\tilde{g}_{33}-\tilde{g}^0_{33}) = \tfrac{1}{2}\nu r^2(1-e^{\alpha}), \\ T_{33} &= T_{22}\sin^2\theta, \\ T_{44} &= \rho\, e^{\beta}. \end{aligned} \tag{3.6}$$

Hence

$$T_1{}^1 = \tfrac{1}{2}(\nu+2\mu)(e^{-\alpha}-1), \tag{3.7}$$

$$T_2{}^2 = T_3{}^3 = \tfrac{1}{2}\nu(1-e^{\alpha}), \tag{3.8}$$

$$T_4{}^4 = -\rho. \tag{3.9}$$

From (2.2)–(2.5) and (3.6)–(3.9), we have

$$e^{-\alpha}\left(\frac{\beta'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = \tfrac{1}{2}(\nu+2\mu)(e^{-\alpha}-1), \tag{3.10}$$

$$e^{-\alpha}\left(\frac{\beta''}{2}-\frac{\alpha'\beta'}{4}+\frac{\beta'^2}{4}+\frac{\beta'-\alpha'}{2r}\right) = \tfrac{1}{2}\nu(1-e^{\alpha}), \tag{3.11}$$

$$e^{-\alpha}\left(\frac{\alpha'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2} = \rho. \tag{3.12}$$

From (3.12) we have

$$e^{-\alpha} = 1-\tfrac{1}{3}r^2\rho+\frac{C_0}{r} \tag{3.13}$$

where C_0 is a constant of integration.

To remove the singularity at the origin we put $C_0 = 0$. Thus

$$e^{-\alpha} = 1-\tfrac{1}{3}r^2\rho. \tag{3.14}$$

From (3.14) and (3.10) we have

$$\beta' = \frac{\tfrac{1}{3}r\rho}{1-\tfrac{1}{3}r^2\rho}+\tfrac{1}{2}(\nu+2\mu)r(1-e^{\alpha})$$

which has a solution of the form

$$e^{\beta} = (1-\tfrac{1}{3}r^2\rho)^{-1/2}\exp\left(\int \tfrac{1}{2}(\nu+2\mu)r(1-e^{\alpha})\,dr\right). \tag{3.15}$$

Hence the Schwarzschild line element in an elastic medium is given by

$$ds^2 = +(1-\tfrac{1}{3}r^2\rho)^{-1}\,dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$
$$-(1-\tfrac{1}{3}r^2\rho)^{-1/2}\exp\left(\int \tfrac{1}{2}(\nu+2\mu)r(1-e^{\alpha})\,dr\right)dt^2. \tag{3.16}$$

In order that the solution may be real we must have $r^2 < 3/\rho$, which puts an upper limit on the possible size of a sphere of given density and on the mass of the sphere of given radius.

In order that equations (3.10)–(3.12) be consistent, they must satisfy the relation

$$G^j_{1;j} = -T^j_{1;j} = 0. \tag{3.17}$$

The left-hand side of the above equation already satisfies this condition. For the right-hand side we have

$$\frac{\partial}{\partial x^j}T_1{}^j+T_1{}^l\Gamma^j_{lj}-T_l{}^j\Gamma^l_{j1} = 0$$

that is,

$$\frac{\partial}{\partial r}(\nu+2\mu)\frac{r^2\rho}{3}+\frac{4}{r}\left(\frac{1}{2}(\nu+2\mu)\frac{r^2\rho}{3}-\frac{1}{2}\nu\frac{\tfrac{1}{3}r^2\rho}{1-\tfrac{1}{3}r^2\rho}\right)+\left(\frac{1}{2}(\nu+2\mu)\frac{r^2\rho}{3}+\rho\right)\frac{\beta'}{2} = 0. \tag{3.17a}$$

This equation determines a relation between μ, ν and ρ which stands for the 'equation of state' for the elasticity tensor (1.7).

The field outside the sphere is empty and is given by Schwarzschild's exterior line element

$$ds^2 = \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2 - \left(1-\frac{2m}{r}\right) dt^2. \tag{3.18}$$

The internal field (3.16) must fit at the boundary $r = a$ with the external field for which we require that g_{ij} and $T_1^{\ 1}$ be continuous at $r = a$ (Synge 1960). We thus have the three conditions

$$m = \tfrac{1}{6}a^2\rho, \tag{3.19}$$

$$\nu(a) + 2\mu(a) = 0, \tag{3.20}$$

and

$$\int_0^a \tfrac{1}{2}r(\nu+2\mu)(1-e^{\alpha})\, dr = \tfrac{3}{2}\ln\left(1-\frac{2m}{a}\right). \tag{3.21}$$

From (3.19) we determine the value of m when the radius and the density are given. Equations (3.20) and (3.21) are two conditions that the function $\nu+2\mu$ has to satisfy at the boundary. When such a function has been chosen the equation (3.17*a*) then uniquely determines the scalars ν and μ.

References

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